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The Multiplicative Jordan Decomposition in Group Rings

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1. INTRODUCTION

Let G be a finite group and K a field of characteristic 0. Then every element α in the group algebra $K[G]$ of G over K has a unique additive Jordan decomposition $\alpha = \alpha_s + \alpha_n$ with $\alpha_s, \alpha_n \in K[G]$, α_s semisimple, α_n nilpotent, and $\alpha_s \alpha_n = \alpha_n \alpha_s$. Recall that an element $\alpha \in K[G]$ is said to be semisimple if the minimal polynomial $m(X)$ of α over K does not have repeated roots in the algebraic closure $\overline{K}$ of K, or, equivalently, if $m(X)$ is coprime to its (formal) derivative $m'(X)$. If the element α is a unit in $K[G]$, then α_s is also a unit and α has a unique multiplicative Jordan decomposition $\alpha = \alpha_s \alpha_u$, with $\alpha_u \in K[G]$ unipotent and $\alpha_s \alpha_u = \alpha_u \alpha_s$. If R is an integral domain with quotient field K and $\alpha \in R[G]$, then α_s need not necessarily lie in $R[G]$. If this happens to be the case for all $\alpha \in R[G]$ (resp. for all units in $R[G]$), then we say that the *additive* (resp. *multiplicative*) *Jordan decomposition* holds in $R[G]$. It is easy to see that if

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the AJD holds in $R[G]$, then so does the MJD. If the unit group $\mathscr{U}(R[G])$ is viewed as a linear group, then the MJD is equivalent to the unit group being a splittable group [3, 13]. Finite groups G for which the AJD holds in the integral group ring $\mathbf{Z}[G]$ have been characterized in [7, 8] and we have given some illustrative examples in [2] for the behavior of the MJD in $\mathbf{Z}[G]$. Our aim in this paper is to explore further the MJD property in integral group rings.

We begin by examining $GL_n(R)$ for an arbitrary integral domain which is integrally closed and not a field. We note that $GL_n(R)$ is splittable for $n = 2$ and is not splittable for $n \geq 4$, while $GL_3(R)$ is splittable if and only if R has the property that the difference of any two distinct units in R is again a unit in R (i.e., the units together with 0 form a subfield). This behavior naturally suggests, as in the case of the AJD, a severe restriction on the degrees of the Wedderburn components of the rational group algebra $\mathbf{Q}[G]$ for the MJD to hold in $\mathbf{Z}[G]$. Indeed, we show that the degrees must all be less than or equal to 3. As a contribution toward characterizing the MJD for integral group rings, we examine several groups and show, in particular, that the MJD holds in the integral group ring $\mathbf{Z}[Q_{4p}]$ of a generalized quaternion group Q_{4p}, and that the only dihedral groups D_{2n} of order $2n$ for which the MJD holds in $\mathbf{Z}[D_{2n}]$ are those with $n = 2$, 4, or an odd prime p.

2. PRELIMINARIES

The decomposition of the elements of a finite group algebra $K[G]$, with K a field of characteristic 0, into semisimple and nilpotent components rests on the following basic result in linear algebra.

PROPOSITION 2.1. *Let V be a finite-dimensional vector space over a field K of characteristic 0 and let $\varphi \in \mathrm{End}_K(V)$. Then there exist unique elements φ_s and $\varphi_n \in \mathrm{End}_K(V)$ such that*

$$\varphi = \varphi_s + \varphi_n, \qquad \varphi_s \varphi_n = \varphi_n \varphi_s,$$

where φ_s is semisimple and φ_n is nilpotent. Moreover, φ_s, φ_n can be expressed as polynomials in φ over K. If φ is invertible, then so is φ_s, and there is a unique factorization

$$\varphi = \varphi_s \varphi_u,$$

with $\varphi_u \in \mathrm{End}_K(V)$ unipotent and $\varphi_s \varphi_u = \varphi_u \varphi_s$.

An analogous result holds for square matrices over K and for elements of $GL_n(K)$.

For the reader's convenience, we recall a method of constructing the semisimple component φ_s for a given linear transformation $\varphi \in \mathrm{End}_K(V)$. (See, for instance, [4], [5], or [12] for more details.) Let $m(X) = \prod_{i=1}^{r} q_i^{e_i}(X)$ be the factorization in $K[X]$ of the minimal polynomial $m(X)$ of φ over K into irreducible factors. Let $g(X) = \prod_{i=1}^{r} q_i(X)$. Then $(g(X), g'(X)) = 1$ and therefore there exist polynomials $h(X)$ and $k(X) \in K[X]$ such that

$$g'(X)h(X) + g(X)k(X) = 1.$$

Choose the least m such that $2^m \geq e_i$ for $i = 1, 2, \dots, r$. Let $\theta\colon K[X] \to K[X]$ be the K-algebra homomorphism given by

$$X \mapsto X - g(X)h(X).$$

Then $\varphi_s = s(\varphi)$, where $s(X) = \theta^m(X)$. In particular, if $\varphi, \psi \in \mathrm{End}_K(V)$ have the same minimal polynomial, then the polynomial $s(X)$ is such that

$$\varphi_s = s(\varphi) \qquad \text{and} \qquad \psi_s = s(\psi).$$

3. GENERAL LINEAR GROUPS

Recall that a subgroup H of $GL_n(K)$ is said to be *splittable* if, for every $A \in H$, the semisimple component A_s of A also lies in H.

Let R be an integral domain of characteristic 0 which is integrally closed in its quotient field K and $R \neq K$. Let $\overline{K}$ denote the algebraic closure of K.

PROPOSITION 3.1. (i) *$GL_2(R)$ is splittable.*

(ii) *$GL_3(R)$ is splittable if and only if, for every $a, b \in \mathscr{U}(R)$, $a \neq b$, $a - b \in \mathscr{U}(R)$.*

(iii) *$GL_n(R)$ is not splittable for $n \geq 4$.*

Proof. Let $A \in GL_n(R)$ and let $f(X)$ be the characteristic polynomial of A. If $f(X)$ has no repeated root, then, by definition, A is semisimple. If $f(X)$ has a root a, say, of multiplicity n, then $a \in R$ and $A_s = aI$. Thus (i) follows. To prove (ii), suppose $f(X) = (X-a)^2(X-b)$, $a, b \in \overline{K}$, $a \neq b$. Then again it is easy to see that $a, b \in \mathscr{U}(R)$. If R has the indicated property, then

$$A_s = A - \frac{(A - aI)(A - bI)}{a - b}.$$

Consideration of the semisimple part of $\begin{pmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & b \end{pmatrix}$, $a, b \in \mathscr{U}(R)$, $a \neq b$, shows that condition (ii) is necessary, for the semisimple part of this matrix is $\begin{pmatrix} a & 0 & -(a-b)^{-1} \\ 0 & a & 1 \\ 0 & 0 & b \end{pmatrix}$. To prove (iii), let $p \neq 0$, $p \neq 0, -4$ be an element of R which is not invertible, and write $x = p + 2$. Consider the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & -x & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -x \end{pmatrix}.$$

The minimal polynomial of A is $(X^2 + xX + 1)^2$ and therefore

$$A_s = A - \frac{1}{x^2 - 4}(A^2 + xA + I)(2A + xI)$$
$$= \begin{pmatrix} 0 & 1 & \dfrac{x}{x^2-4} & \dfrac{x^2-2}{x^2-4} \\ -1 & -x & \dfrac{-2}{x^2-4} & \dfrac{-x}{x^2-4} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -x \end{pmatrix},$$

which does not belong to $GL_4(R)$ since $p = x - 2$ is not invertible in R. Finally, note that if $A \in GL_n(R)$ and $B = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$, then $B_s = \begin{pmatrix} A_s & 0 \\ 0 & 1 \end{pmatrix}$ and therefore if $GL_n(R)$ is not splittable, then $GL_r(R)$ is not splittable for all $r \geq n$. This completes the proof.

Remark 3.2. Let $M_n(R)$ denote the ring of $n \times n$ matrices over R. The preceding proof shows that the AJD holds in $M_n(R)$ if $n = 2$ and fails for $n \geq 3$; that is, if $A \in M_2(R)$, then $A_s \in M_2(R)$ and, for $n \geq 3$, there exist matrices over R whose semisimple parts do not have all their entries in R. Note that for the AJD to hold in $M_2(R)$ one only requires that, whenever 2α and α^2 are in R for α in K, then α is in R—while for $n \geq 3$, the AJD fails for every integral domain which is not a field.

The preceding result immediately yields the following characterization for the AJD and the MJD to hold in $R[G]$ when the ring R of coefficients is a ring $K[X]$ of polynomials over a field K for which $K[G]$ is a direct sum of matrix rings over fields.

THEOREM 3.3. *If G is a finite group and K is a field of characteristic* 0 *such that*

$$K[G] \simeq \bigoplus_{i=1}^{h} M_{n_i}(K_i),$$

where the K_i's are field extensions of K, then

(i) *$K[X][G]$ has the AJD if and only if $n_i \leq 2$ for $i = 1, 2, \ldots, h$;*

(ii) *$K[X][G]$ has the MJD if and only if $n_i \leq 3$ for $i = 1, 2, \ldots, h$.*

4. IRREDUCIBLE DEGREES

THEOREM 4.1. *Let G be a finite group and R an integral domain of characteristic* 0 *which is not a field. Then, for the multiplicative Jordan decomposition to hold in $R[G]$, it is necessary that the degrees of the Wedderburn components of $K[G]$, where K is the field of fractions of R, must all be less than or equal to* 3.

Proof. Let $K[G] \simeq \oplus_{i=1}^{h} M_{n_i}(D_i)$ be the Wedderburn decomposition of $K[G]$ and assume that $n_1 \geq 4$. Let π_i: $K[G] \to M_{n_i}(D_i)$ be the natural projections, $i = 1, 2, \ldots, h$. Then we can choose nilpotent elements $x, y \in R[G]$ and $a, b \in R$ such that $ab \neq 0$,

$$\pi_1(x) = \begin{pmatrix} 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \pi_1(y) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 \end{pmatrix},$$

$\pi_i(x) = 0 = \pi_i(y)$ for all $i \geq 2$. Let $r(\neq 0) \in R$, with $rab \neq -4$. Consider the elements $u_1 = 1 + x$, $u_2 = 1 + ry$, both of which are units and therefore so is their product $u = u_1 u_2$. Now $\pi_1(u) = I + A$, where

$$A = \begin{pmatrix} 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & rab & a \\ 0 & 0 & rb & 0 \end{pmatrix}.$$

The semisimple component of A is

$$A_s = \begin{pmatrix} 0 & 0 & 0 & a^2/rb \\ 0 & 0 & a & 0 \\ 0 & 0 & rab & a \\ 0 & 0 & rb & 0 \end{pmatrix}.$$

If u has a Jordan decomposition in $R[G]$, then $\pi_1(u_s) \in \pi_1(R[G])$ and so a^2/rb lies in $S = \pi_1(R[G])_{14}$, the finitely generated R-submodule of D_1 consisting of the $(1,4)$-entries of the elements in $\pi_1(R[G])$. Hence S contains all of K. Let $x_1, x_2, \dots, x_n$ be a set of generators for S over R. Choose a basis $\{y_\lambda\}$ of D_1 over K containing 1, say with $y_1 = 1$. Then we have $x_i = \sum_\lambda c_{i,\lambda} y_\lambda$, with $c_{i,\lambda} \in K$. For each $k \in K \subset S$ we have $k = \sum_i r_i x_i = \sum_i r_i \sum_\lambda c_{i,\lambda} y_\lambda = \sum_\lambda (\sum_i r_i c_{i,\lambda}) y_\lambda$. Only the $\lambda = 1$ term can be nonzero and we have $k = \sum_i r_i c_{i,1}$. Thus K is finitely generated as an R-module and hence $K = R$, a contradiction, and the result is proved.

Note that the latter part of the preceding argument allows us, in Theorem 3.1 of [7], to only assume that R is an integral domain which is not a field. An implicit corollary of Theorem 3.1 of [7] is that, for groups G of odd order, $R[G]$ has the AJD if and only if G is abelian. The multiplicative analogue of this is as follows.

COROLLARY 4.2. *If neither* 2 *nor* 3 *divides the order of* G, *then* $R[G]$ *has the MJD if and only if* G *is abelian.*

5. NONSPLITTABLE UNIT GROUPS

Let p, q be odd primes and let C_n, D_{2n} denote the cyclic group of order n and the dihedral group of order $2n$, respectively. We will determine which D_{2n} have the MJD for their integral group rings.

PROPOSITION 5.1. *If G is any one of the following groups, then its integral group ring does not have the multiplicative Jordan decomposition*:

(i) $G = \langle x, t \mid x^{pq} = t^{2^k} = 1,\ txt^{-1} = x^{-1} \rangle$.

(ii) $G = \langle x, y, t \mid x^p = y^p = t^{2^k} = 1,\ xy = yx,\ txt^{-1} = x^{-1},\ tyt^{-1} = y^{-1} \rangle$.

(iii) $G = \langle x, y, t \mid x^p = y^q = t^{2^k} = 1,\ xy = yx,\ yt = ty,\ txt^{-1} = x^{-1} \rangle$.

(iv) $G = \langle x, t \mid x^{2p} = t^{2^k} = 1,\ txt^{-1} = x^{-1} \rangle$.

(v) $G = \langle x, t \mid x^8 = t^{2^k} = 1,\ txt^{-1} = x^{-1} \rangle$.

(vi) $G = \langle x, t \mid x^p = t^{2^{k+1}} = 1,\ txt^{-1} = x^e \rangle$ *where the multiplicative order of e mod p is* 4 (*hence* 4 *must divide* $p - 1$).

In each case k is an arbitrary positive integer.

Proof. We will exhibit, for each group listed in (i)–(v), a unit u in its integral group ring whose nilpotent component u_n does not have integral coefficients. The verification in each case is straightforward, and is therefore omitted. Such verification is aided by applying the standard representation of G in $M_2(\mathbf{Z}[H])$ where H is a normal abelian subgroup of G of index 2 (see, e.g., the proof of Proposition 3.1 of [8]).

(i) $p \neq q$:

$$u = x^p\left(1 + (x^q - x^{-q})(1 + t^2 + \cdots + t^{2^k-2})(1+t)\right),$$

$$u_n = \frac{1}{q}(1 + x^p + \cdots + x^{p(q-1)})(x^q - x^{-q})(1 + t^2 + \cdots + t^{2^k-2})(1+t).$$

$p = q$:

$$u = x^p\left(1 + (x - x^{-1})(1 + t^2 + \cdots + t^{2^k-2})(1+t)\right),$$

$$u_n = \frac{1}{p}(1 + x^p + \cdots + x^{p(p-1)})(x - x^{-1})(1 + t^2 + \cdots + t^{2^k-2})(1+t).$$

(ii)

$$u = y\left(1 + 2(x - x^{-1})(1 + t^2 + \cdots + t^{2^k-2})(1+t)\right),$$

$$u_n = \frac{2}{p}(x - x^{-1})(1 + y + \cdots + y^{p-1})(1 + t^2 + \cdots + t^{2^k-2})(1+t).$$

(iii)

$$\begin{aligned} u = {} & \left[1 + (x - x^{-1})(1 + t^2 + \cdots + t^{2^k-2})(1+t)\right] \\ & \times\left[\left(1 + (x - x^{-1})(1 + t^2 + \cdots + t^{2^k-2})(1-t)\right) \right. \\ & \qquad \left. \times (1 + y + \cdots + y^{q-1})\right], \end{aligned}$$

$$\begin{aligned} u_n = {} & (x - x^{-1})(1 + t^2 + \cdots + t^{2^k-2})(1+t) \\ & \times\left(1 - \frac{1}{q}(1 + y + \cdots + y^{q-1})\right). \end{aligned}$$

(iv)

$$\begin{aligned} u = {} & \left[1 + (x^2 - x^{-2})(1 + t^2 + \cdots + t^{2^k-2})(1+t)\right] \\ & \times\left[1 - (x^2 - x^{-2})(1 - x^p)(1 + t^2 + \cdots + t^{2^k-2})(1-t)\right], \end{aligned}$$

$$u_n = \frac{1}{2}(x^2 - x^{-2})(1 + x^p)(1 + t^2 + \cdots + t^{2^k-2})(1+t).$$

(v)

$$u = x^2\left(1 + (x + 2x^3 - 2x^5 - x^7)(1 + t^2 + \cdots + t^{2^k-2})(1+t)\right),$$

$$u_n = \frac{1}{2}(x - x^3 + x^5 - x^7)(1 + t^2 + \cdots + t^{2^k-2})(1+t).$$

(vi) When $k = 1$ the group ring $\mathbf{Q}[G]$ has an irreducible component of degree 4 (see [8, p. 24]). The corresponding group ring for arbitrary k has the $k = 1$ case as homomorphic image so will also have such a component. Hence Theorem 4.1 applies.

It has been shown in [2] that the MJD holds in the integral group ring of the dihedral group D_8. The $k = 1$ case of the preceding proposition therefore immediately gives a characterization of the dihedral groups D_{2n} for which $\mathbf{Z}[D_{2n}]$ has the MJD.

THEOREM 5.2. *The multiplicative Jordan decomposition holds in* $\mathbf{Z}[D_{2n}]$ *if and only if* $n = 2, 4,$ *or an odd prime*.

Note that Proposition 5.1, for $k = 1$, shows that the MJD fails for each of the groups D_{2pq}, $(C_p \times C_p) \rtimes C_2$ (where the generator of C_2 inverts each element of $C_p \times C_p$), $C_q \times D_{2p}$, D_{4p}, D_{16}, and $C_p \rtimes C_4$ (where C_4 acts faithfully on C_p). These include all but one of the groups whose integral group rings were shown not to have the AJD in Proposition 3.1 of [8]. The one missing is the generalized quaternion group dealt with below.

6. GENERALIZED QUATERNION GROUPS

Again let p be an odd prime.

THEOREM 6.1. *The multiplicative Jordan decomposition holds in the integral group ring of* $Q_{4p} = \langle x, t \mid x^p = t^4 = 1,\ txt^{-1} = x^{-1} \rangle$, *the group of generalized quaternions of order* $4p$.

Proof. Consider the complex representations of Q_{4p} given by

$$\rho_1\colon x \mapsto 1, \qquad t \mapsto 1,$$

$$\rho_2\colon x \mapsto 1, \qquad t \mapsto -1,$$

$$\rho_3\colon x \mapsto 1, \qquad t \mapsto i,$$

$$\rho_4\colon x \mapsto \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\rho_5\colon x \mapsto \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

where i and ζ are primitive fourth and pth roots of unity, respectively. From [6] we have

$$\mathbf{Q}[Q_{4p}] \simeq \mathbf{Q} \oplus \mathbf{Q} \oplus \mathbf{Q}(i) \oplus \rho_4(\mathbf{Q}[Q_{4p}]) \oplus \rho_5(\mathbf{Q}[Q_{4p}]).$$

Let

$$u = \sum_{i=0}^{p-1} \alpha_i x^i + \sum_{i=0}^{p-1} \beta_i x^i t + \sum_{i=0}^{p-1} \gamma_i x^i t^2 + \sum_{i=0}^{p-1} \delta_i x^i t^3,$$

where $\alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbf{Z}$, be an arbitrary unit in $\mathbf{Z}[Q_{4p}]$. With the help of the complex representations ρ_i of Q_{4p}, one can easily show that if u is not semisimple then the nilpotent component of u is

$$u_n = \Bigg[\frac{1}{4}\sum_{i=1}^{(p-1)/2} \big((\alpha_i + \gamma_i) - (\alpha_{p-i} + \gamma_{p-i})\big)(x^i - x^{-i}) + \frac{1}{2p}\sum_{i=0}^{p-1} \big(p(\beta_i + \delta_i) - (\beta + \delta)\big)x^i t\Bigg](1 + t^2),$$

where $\beta = \beta_0 + \beta_1 + \cdots + \beta_{p-1}$ and $\delta = \delta_0 + \delta_1 + \cdots + \delta_{p-1}$. By considering the images of u mod $\langle t^2 \rangle$ and mod $\langle x \rangle$, it can further be verified that $u_n \in \mathbf{Z}[Q_{4p}]$. This involves an application of Lemma 5 of [9] to the matrix $\rho_4(u)$ to deduce that $\beta_i = \delta_i$ for all i, and then working with $\rho_5(u_n)$.

We can also show that the integral group ring of a certain group of order 16, with presentation similar to that of Q_{4p}, has the MJD.

THEOREM 6.2. *The multiplicative Jordan decomposition holds in the integral group ring of* $G = \langle x, t \mid x^4 = t^4 = 1, txt^{-1} = x^{-1} \rangle$.

Proof. The group G has as homomorphic images both $C_4 \times C_2$ (by mapping x^2 to the identity) and the quaternion group Q_8 (by identifying x^2 and t^2). It is known [11, Theorem 2.7] that, for each of these latter groups, every unit in the integral group ring is of the form $+g$ or $-g$ for some group element g. Furthermore, we can multiply any unit u in $\mathbf{Z}[G]$ by one of the central units $1, x^2, t^2, x^2t^2$ without affecting its Jordan decomposability. This allows us to assume without loss of generality that (up to sign) every unit in $\mathbf{Z}[G]$ is of the form $u = v + (a + bx + ct + dxt)(1 - x^2)(1 + t^2)$ with a, b, c, d integers and v one of the elements $1, x, t, xt$. We now apply the previously mentioned standard representation of G into $M_2(\mathbf{Z}[H])$, where $H = \langle x, t^2 \rangle$, to conclude that u will be semisimple if v is either x, t, or xt. When v is 1 we also conclude that u is semisimple unless $a = 0$ and $b^2 = c^2 + d^2$. Finally, when these conditions hold, we find that $u_n = (bx + ct + dxt)(1 - x^2)(1 + t^2)$, completing the proof since this has integral coefficients.

7. REMARKS

Remark 1. By Theorem 4.1, given a finite group G, for the MJD to hold in $\mathbf{Z}[G]$ the degrees of the Wedderburn components of $\mathbf{Q}[G]$ must all be less than or equal to 3. The case when the degrees are all 1 is well understood (see [1, 10]). If the degrees are all less than or equal to 2 and the Sylow 2-subgroups of G are abelian, then, proceeding as in [8], we see that, in view of Proposition 5.1, G must be of the type $C_p \rtimes C_{2^m}$ with p an odd prime. To complete the characterization in this case, one needs to consider in the first instance such groups and the finite 2-groups.

Remark 2. Given a finite group G, let S (resp. U) be the set of all semisimple (resp. unipotent) units in $\mathbf{Z}[G]$. Then the subgroup $Y = \langle S, U \rangle$ generated by S and U is a normal subgroup and $\mathscr{U}(\mathbf{Z}[G])/Y$ is a finitely generated (since $\mathscr{U}(\mathbf{Z}[G])$ is so) and periodic group. The periodicity is a consequence of the fact that every element $u \in \mathscr{U}(\mathbf{Z}[G])$ has a power u^t with its Jordan components in $\mathscr{U}(\mathbf{Z}[G])$. It might be of interest to examine this quotient further.

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